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CHAOS SYNCHRONIZATION OF TWO PARAMETRICALLY EXCITED PENDULUMS

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Two desynchronous parametrically excited chaos pendulums demonstrate a type of hyperchaotic behavior. This paper presents a periodic feedback scheme to synchronize the two pendulum subsystems. The synchronization principle is investigated. Feedback control parameters are discussed. The synchronous parameter intervals in which the maximum transverse Lyapunov exponent is negative guarantee the achievement of synchronization. Two originally unrelated tumbling chaos pendulums can be synchronized by this method in the intervals.

1. INTRODUCTION

The parametrically excited pendulum is used to model the behavior of many engineering systems, such as offshore platforms, buildings in earthquakes etc. Its study has been widely studied [1–3]. Many complex phenomena of this kind of non-linear dynamic system have been demonstrated. The different chaotic motions and their respective stable zones in parameter space have been described in numerical calculations and experiments, which lead to a rich field for the research of parametrically excited pendulums. However, when the synchronous motion of two uncoupled pendulums with different initial conditions is considered, such chaotic dynamic behavior exhibits the instinct difficulty. Known as the "butterfly effect" [4], the sensitive dependence on initial conditions means that any error of initial conditions increases exponentially and leads to a remarkably different result. The pseudo random motion of multiple pendulums demonstrates extremely complex hyperchaotic behavior. How to synchronize more than one chaos pendulums becomes puzzling and interesting.

Since Pecora and Carroll proposed their method of synchronizing chaos [5], theoretical as well as experimental research has been carried out in a variety of non-linear dynamic systems. Chaos synchronization makes it possible to synchronize two chaotic systems previously considered impossible. It has widely aroused research in the light of potential applications [6–8]. In this paper, the synchronization of tumbling chaos [1] of two parametrically excited pendulums by the feedback synchronization method is realized. Here, the feedback is

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designed to periodically impose on one pendulum. With the values of the normalized periodic factor and the feedback weight in the synchronous parameter intervals, the "parametrically excited hyperchaos pendulums system" (PEHPS) consisting of two unrelated tumbling chaotic motions will consistently converge to a single chaotic behavior. By chaos synchronization, one brings two chaos pendulums into step. What follows is the discussion of the synchronous parameter intervals.

2. SYNCHRONIZATION PRINCIPLE

The systems under consideration are illustrated in Figure 1, where pendulums A, B correspond to the drive system and the response system respectively. They are connected through the variable feedback. The normalized dynamics of the systems can be described as:

drive pendulum:

$$\ddot{\theta}_{\mathrm{d}} + \beta\dot{\theta}_{\mathrm{d}} + (1 + p\cos(\omega t))\sin(\theta_{\mathrm{d}}) = 0, \tag{1}$$

response pendulum:

$$\ddot{\theta}_r + \beta\dot{\theta}_r + (1 + p\cos(\omega t))\sin(\theta_r) = \mathrm{K(t)}, \tag{2}$$

and

$$K(t) = \left\{ \begin{array}{ll} 0, & t_0 + n(\tau_1 + \tau_2) < t < t_0 + n(\tau_1 + \tau_2) + \tau_1, \quad n = 0, 1 \cdots \\ \mathbf{MF}(\theta_{\mathrm{d}} - \theta_r, \dot{\theta}_{\mathrm{d}} - \dot{\theta}_r), & t_0 + n(\tau_1 + \tau_2) + \tau_1 < t < t_0 + (n+1)(\tau_1 + \tau_2) \end{array} \right\}, \tag{3}$$

where t_0 is the initial time, and θ_{d}, θ_{r} are the angular displacements of the drive and response systems respectively. The periodic force $p\ \cos(\omega t)$ and the damping coefficient β are selected to represent the same experimental conditions of two pendulums. $K(t)$ is the feedback control function. $\mathbf{M}$ is feedback weight matrix. $\mathbf{F}(\theta_{\mathrm{d}} - \theta_{\mathrm{r}},\ \dot{\theta}_d - \dot{\theta}_r)$ is feedback function matrix. Here, the feedback is added periodically. τ_1 denotes the interval where two pendulums are uncoupled while τ_2 denotes the interval where feedback is active. In order to discuss the

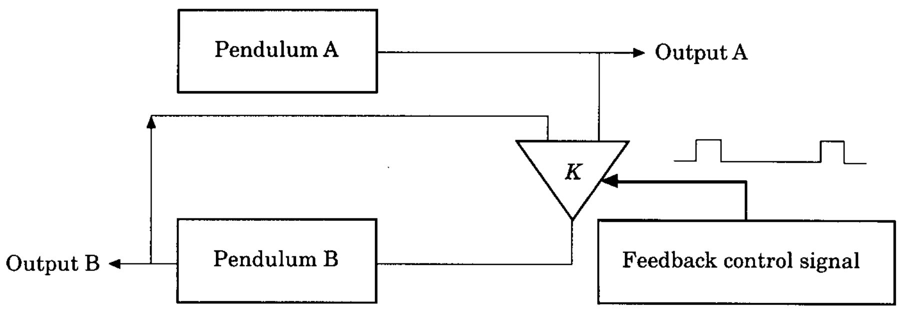

Figure 1. The Schematic diagram of synchronizing parametrically excited pendulums by the periodic feedback synchronization scheme.

synchronization principle generally, one defines the normalized periodic factor as $D = (\tau_1 + \tau_2)/\tau_2$.

It has been found that the values of the parameters p and ω can lead to a number of different types of dynamic behavior for a single pendulum [3]. Within the parameter zone $p = 2$, $\omega = 2$, $\beta = 0{\cdot}1$ of tumbling chaos, the pendulum demonstrates a kind of irregular clockwise rotation and random changing in the rotational direction. The phase portrait associated with this kind of motion is shown in Figure 2(a) and the chaotic attractor is given in Figure 2(b). Figure 2(c) shows the noise-like power spectrum of tumbling chaos of the drive pendulum, where some resonant frequency components embedded in a broadband "noise" spectrum can be seen. By the calculation of the time evolution of the Lyapunov exponents in Figure 2(d), one can describe quantitatively the dynamical behavior of the pendulum. Lyapunov exponents describe the average exponent rate of divergence or convergence of perturbations in phase space. Any system containing at least one positive Lyapunov exponent is defined to be chaotic. Here the temporal convergence of the maximum Lyapunov exponent $\lambda_1 = 0{\cdot}25$ means that the "random" rotation and oscillation behavior of the pendulum is tumbling chaos. At this time, if the feedback weight $M = 0$, then any uncertainty of the initial conditions of two uncoupled pendulums will be amplified exponentially. The angle divergence makes it impossible to synchronize the tumbling chaos motions of two pendulums. Figure 3(a) shows the time evolution of the angle displacements θ_r, θ_d, where the slight different initial conditions have been taken as $\theta_r(0) = 0{\cdot}01$, $\theta_d(0) = 0{\cdot}011$, $\dot{\theta}_r(0) = 0, \dot{\theta}_d(0) = 0$. However, after several drive periods $T = 2\pi/\omega$, they lead to the remarkably different results. Meanwhile, since two tumbling chaos pendulums cannot be synchronized, there must exist two positive Lyapunov exponents. This pendulum system with two positive Lyapunov exponents was called the "parametrically excited hyperchaos pendulums system" (PEHPS). Figures 3(b) and (c) show the complex behavior of this kind of hyperchaotic attractor.

Chaos synchronization means that if the trajectories are located in the synchronization manifold, trajectories of the chaotic systems (2) will converge to the attractor of equation (1) and $\theta_r = \theta_d$, $\dot{\theta}_r = \dot{\theta}_d$. In order to synchronize two chaos pendulums, one considers the feedback synchronization method. The right-side of equation (2) is driven by the feedback control function $K(t)$. With the coupled systems, the spectrum of Lyapunov exponents can be divided into two subsets λ_i^s and $\lambda_i^s (i = 1, 2)$, which are within the synchronization manifold and transverse to synchronization manifold respectively [9]. The first one describing the evolution of perturbations along the synchronization manifold is associated with the drive pendulum. The second one describes the perturbation transverse to the synchronization manifold. If two chaotic systems are not synchronous, there is at least one positive transverse Lyapunov exponent. Then any slight perturbations along the transverse direction will be quickly amplified so that the limit set is no longer restricted to the synchronization manifold, and trajectories will wander in a high-dimensional phase space of the hyperchaotic

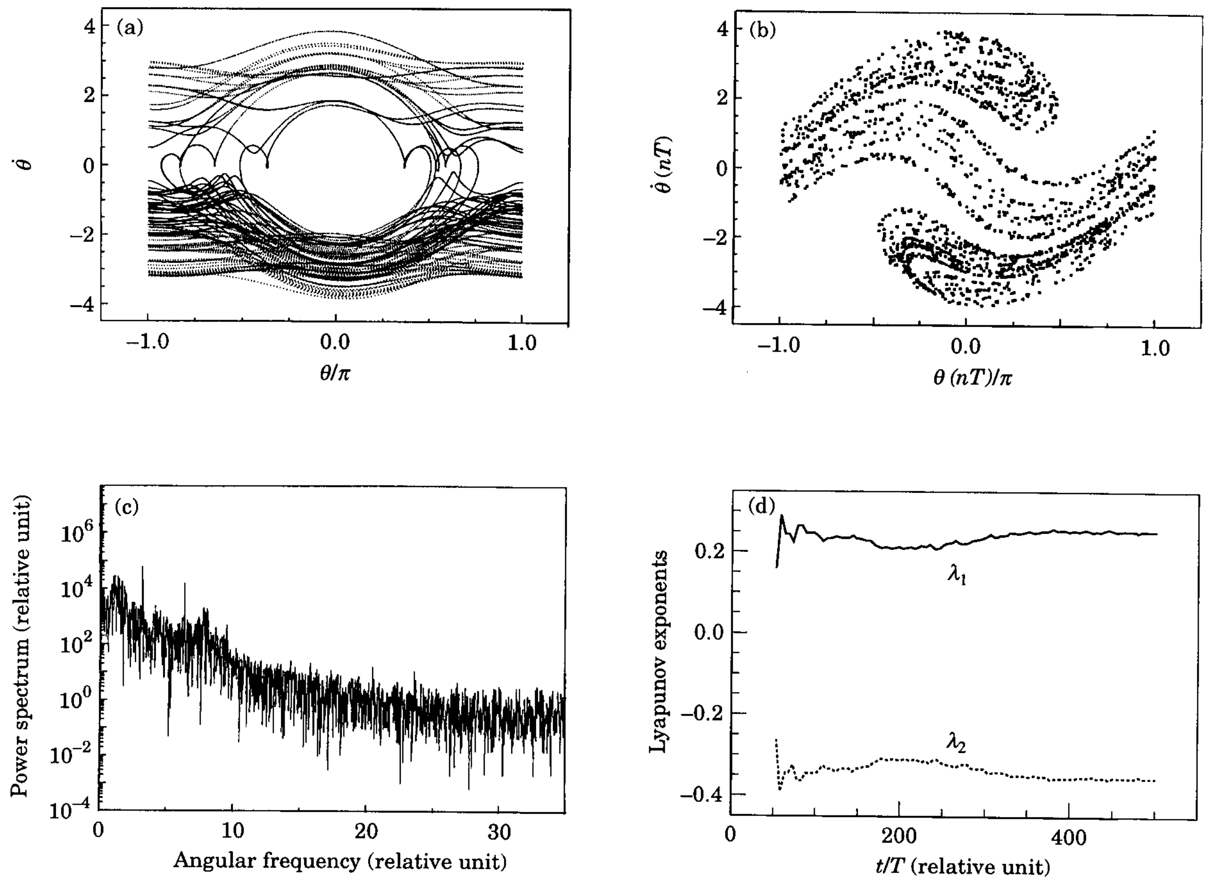

Figure 2. The tumbling chaos of the drive pendulum. (a) The phase portrait, θ/π versus $\dot{\theta}$; (b) the Poincaré section of the chaotic attractor, $\theta(nT)/\pi$ versus $\dot{\theta}(nT)$ ($n = 0, 1 \ldots$); (c) the power spectrum of the drive pendulum; (d) the time evolution of the Lyapunov exponents. λ_1 and λ_2 identified in the figure correspond to the first (or the maximum) and the second Lyapunov exponent respectively.

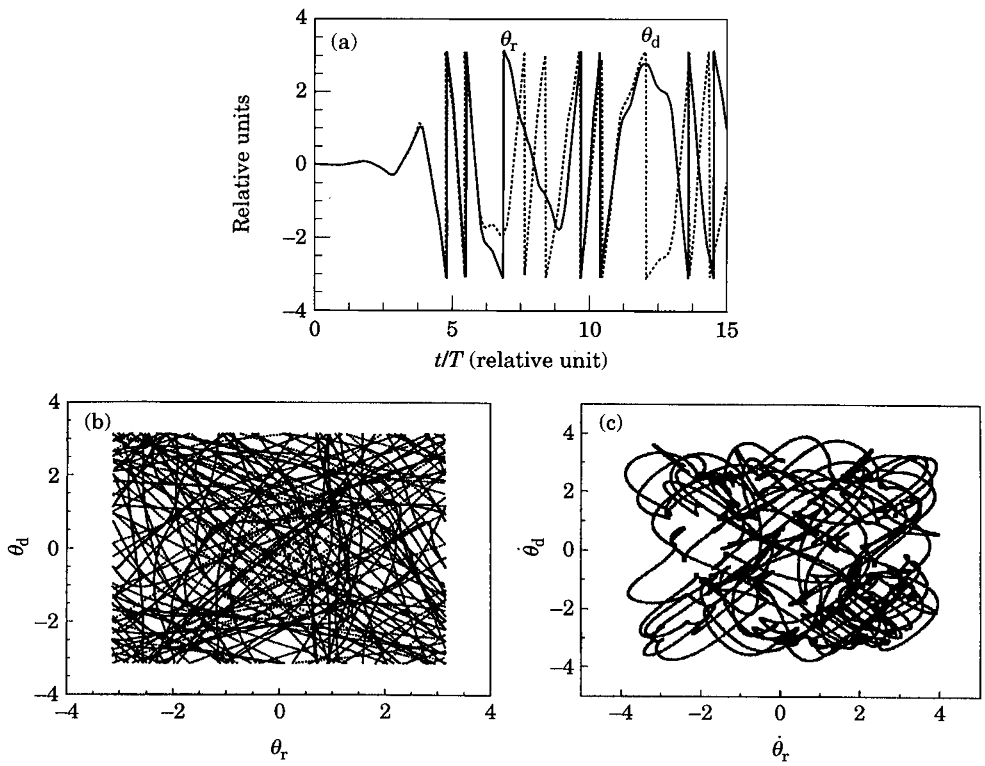

Figure 3. Parametrically excited hyperchaos pendulums system (PEHPS). (a) the time evolution of the angle displacements θ_r, θ_d, where the slightly different initial conditions have been taken as $\theta_r(0) = 0{\cdot}01$, $\theta_d(0) = 0{\cdot}011$, $\dot{\theta}_r(0) = 0$, $\dot{\theta}_d(0) = 0$; (b) the phase portrait θ_r versus θ_d of the hyperchaotic attractor; (c) the phase portrait θ_r versus θ_d of the hyperchaotic attractor. Key: —, θ_r; $\cdots\cdot\theta_d$.

attractor. Thus the synchronization problem is to let the maximum transverse Lyapunov exponent $\lambda^t_{max} < 0$.

Since feedback control function is arbitrary, it can lead to many types of feedback schemes of chaos synchronization. Pyragas had proposed the method of small time continuous feedback [10]. The feedback control function $K(t)$ was taken as the continuous function. However, with discontinuous feedback, how to synchronize chaos is also interesting. In this paper, a periodic feedback scheme is suggested. In some intervals, feedback is active; while in some other intervals two chaotic systems are uncoupled. The normalized periodic factor D is variable. Feedback weight and function matrix are selected to be

$$\mathbf{M} = (M_1, M_2) \quad \text{and} \quad \mathbf{F}(\theta_d - \theta_r, \dot{\theta}_d - \dot{\theta}_r) = \begin{pmatrix} \theta_d - \theta_r \\ \dot{\theta}_d - \dot{\theta}_r \end{pmatrix}$$

respectively. The response chaos pendulum is driven by the feedback control force that is proportional to the differences of the angular displacements or angular velocities of two pendulums. With the values of the feedback weight as well as the normalized factor D in the synchronous parameter intervals, the maximum transverse Lyapunov exponent can be reduced to a negative value.

Synchronization can then be achieved. Two pendulums in the post-transient regime will move synchronously as if there were only a single parametrically excited pendulum driven by an external control force. Figure 4(a) shows the time evolution of $(\Delta Z)^2 = (\theta_r - \theta_d)^2 + (\dot{\theta}_r - \dot{\theta}_d)^2$, where $D = 10$, e.g., feedback is active at every tenth sample and the feedback weight $(M_1, M_2) = (0, 0)$ and $(0, 10)$ respectively. When $(M_1, M_2) = (0, 0)$, i.e., the feedback weight has not been imposed on the responser in interval *a*, then the slight perturbation to initial conditions is amplified to a large distortion. The two pendulums move independently and the motion wanders in a high-dimensional phase space of a hyperchaotic attractor. But when $t > 25T$, after switching on the feedback $(M_1, M_2) = (0, 10)$ in interval *b*, the difference will decrease to a very small value after the transient process despite the feedback being imposed at every tenth sample. The hyperchaotic attractor is constrained into a low-dimensional attractor. The line θ_r, θ_d in Figure 4(b) shows the synchronization of two chaos pendulums where the maximum transverse Lyapunov exponent $\lambda^t_{max} = -0{\cdot}22$.

3. SYNCHRONOUS PARAMETER INTERVALS

The synchronization principle demonstrates that the negative maximum transverse Lyapunov exponent guarantees the synchronization of two coupled chaotic systems. It determines the synchronous parameter zone in the feedback parameter space *M–D*. Certainly, the larger the area of the synchronous zone is, the more stable and efficient the synchronizing chaos pendulums will become. As an example, two types of special synchronous interval in the parameter space are discussed. Figure 5(a) shows the dependence of the maximum transverse Lyapunov exponent λ^t_{max} on the feedback weight *M* where $D = 10$, and those curves *a*, *b*, *c* correspond to $(M_1, M_2) = (M, 0)$, $(0, M)$, (M, M) respectively. The range $M > M_{min}$ in which λ^t_{max} is negative defines the interval of *M* where synchronization can be achieved and $(\theta_r - \theta_d)^2 + (\dot{\theta}_r - \dot{\theta}_d)^2 \to 0$. Here, the scope of synchronous interval of *M* being in the order of curves *a*, *b*, *c* means that the multivariable feedback control in interval *c* leads to the most negative λ^t_{max} so that one can perform fast and stable chaos synchronization. Also the

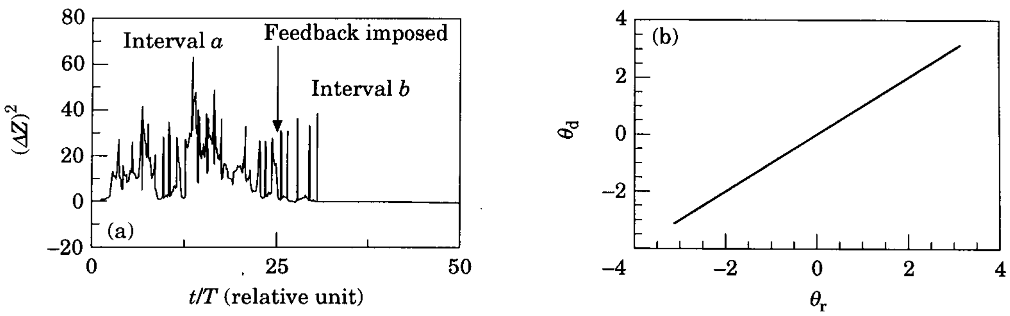

Figure 4. Synchronizing two chaos pendulums. (a) the time evolution of $(\Delta Z)^2 = (\theta_r - \theta_d)^2 + (\dot{\theta}_r - \dot{\theta}_d)^2$ where $D = 10$, $\tau_1 = 9H$, $\tau_2 = 1H$, and $H = T/200$ denotes the time step of calculation. The feedback weights $(M_1, M_2) = (0, 0)$, $(0, 10)$ correspond to intervals *a*, *b*, respectively; (b) the synchronization line θ_r, θ_d.

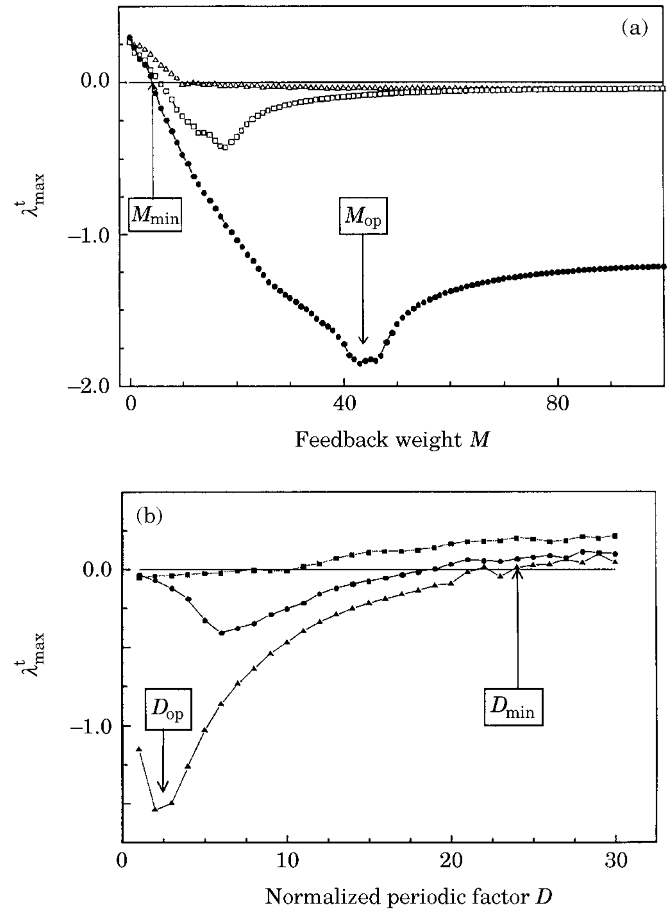

Figure 5. Synchronous parameter intervals. (a) the dependence of the maximum transverse Lyapunov exponent λ^t_{max} on the feedback weight M with the normalized periodic factor $D = 10$; Key: △△△, a; ○○○, b; ●●●, c. (b) the relation between λ^t_{max} and the factor D with the fixed feedback weight $M = 10$. The curves a, b, c correspond to $(M_1, M_2) = (M, 0)$, $(0, M)$ and (M, M), respectively. The values M_{min}, M_{op}, D_{op}, D_{max} of curve c are identified by arrows. Key: a; b; c.

synchronization in interval a is less stable since λ^t_{max} approaches zero. In the synchronous intervals, $\lambda^t_{max}(M)$ has the minimum at $M = M_{op}$ which leads to the maximal synchronous rate, i.e., the optimal synchronization. When $M < M_{op}$, the convergence of θ associated with $\dot{\theta}$ increases with the increase of feedback weight so that the synchronous rate increases. However, when $M > M_{op}$, the convergence of $\dot{\theta}$ is incongruous with θ for the non-symmetrical feedback control. The mismatch leads to the decrease in the synchronous rate with the increase of feedback weight. Therefore there exists an optimal synchronous rate with $M = M_{op}$. The normalized periodic factor D is also an important factor to influence synchronization. Figure 5(b) shows the relation between λ^t_{max} and the factor D with the fixed feedback weight $M = 10$. The relation $D < D_{max}$ determines the synchronous intervals of D associated with $\lambda^t_{max} < 0$. In particular, one notes that with $D = (\tau_1 + \tau_2)/\tau_2 = 1$ $(\tau_1 = 0)$, the periodic feedback scheme reduces to the method of small time continuous feedback. Also the optimal normalized periodic factor D_{op} can be found in the synchronous intervals where the convergence of variables matches one another. However, when the factor D is changed above the threshold value D_{max} or the value of M is taken out of the interval $M > M_{min}$, the external feedback force cannot suppress the amplification of differences, and thus synchronization fails. After the transient process, the two tumbling chaos pendulums randomly rotate and oscillate about the hanging position. PEHPS is reproduced.

4. CONCLUSIONS

In this paper, the synchronization of tumbling chaos of two parametrically excited pendulums by the periodic feedback synchronization method has been realized. The synchronization principle has been investigated. To guarantee synchronization, the feedback control parameter space was discussed. When the values of the normalized periodic factor and the feedback weight are selected in the synchronous parameter zone, two disorder and desynchronous tumbling chaotic pendulums constructing the PEHPS will consistently converge to a single chaotic behavior. The motion in step of two chaos pendulums has been achieved.

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REFERENCES

1. M. J. Clifford and S. R. Bishop 1994 *Journal of Sound and Vibration* **172**, 572–576. Approximating the escape zone for the parametrically excited pendulum.
2. R. W. Leven, B. Pompe, C. Wilke and B. P. Koch 1985 *Physica D* **16**, 371–384. Experiments on periodic and chaotic motions of a parametrically forced pendulum.
3. S. R. Bishop and M. J. Clifford 1996 *Journal of Sound and Vibration* **189**, 142–147. Zones of chaotic behavior in the parametrically excited pendulum.
4. E. N. Lorenz 1963 *Journal of the Atmospheric Sciences* **20**, 130–141. Deterministic nonperiodic flow.
5. L. M. Pecora and T. L. Carroll 1990 *Physical Review Letters* **64**, 821–823, Synchronization in chaotic system.
6. K. M. Cuomo and A. V. Oppenheim 1993 *Physical Review Letters* **71**, 65–68. Circuit implementation of synchronized chaos with application to communications.
7. L. Kocarev and U. Parlitz 1995 *Physical Review Letters* **74**, 5028–5031. General approach for chaotic synchronization with application to communication.
8. Y. Zhang, M. Dai, Y. M. Hua, W. S. Ni and G. H. Du 1998 *Physical Review E* **58**, 3022–3207. Digital communication by active-passive-decomposition synchronization in hyperchaotic systems.
9. J. F. Heagy, T. L. Carroll and L. M. Pecora 1994 *Physical Review E* **50**, 1874–1885. Synchronous chaos in coupled oscillator systems.
10. K. Pyragas 1993 *Physics Letters A* **181**, 203–210. Predictable chaos in slightly perturbed unpredictable chaotic systems.